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Ding, Tao; Bo, Rui; Yang, Yongheng; Blaabjerg, Frede

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Impact of Negative Reactance on Definiteness of B-Matrix and Feasibility of DC Power Flow

Tao Ding, *Member, IEEE*, Rui Bo, *Senior Member, IEEE*
Yongheng Yang, *Member, IEEE*, Frede Blaabjerg, *Fellow, IEEE*

Abstract—This paper reports an essential phenomenon on the existence of “negative reactance” in practical power system models. The negative reactance issue is important, as it could affect the definiteness of the B admittance matrix of power networks and the feasibility of DC power flow. With the graph theory, the B matrix can be treated as a Laplacian matrix. Several theorems and corollaries are given with proof to study the definiteness of the Laplacian matrix with negative weights. Based upon these theorems, the exploration in this paper demonstrates that the negative reactance may even result in “physical dis-connectivity” and make the linear system singular, so that the DC power flow will be infeasible. The results on several test systems show that the location and value of the negative reactance affect the DC power flow feasibility.

Index Terms— Negative reactance, graph theory, Laplacian matrix, DC power flow, eigenvalue

NOMENCLATURE

\mathcal{G}	A graph
$\mathcal{G}_+ / \mathcal{G}_-$	Positive and negative sub-graphs of graph \mathcal{G}
$\mathcal{G}_{\tau+} / \mathcal{G}_k$	Different panning trees of sub-graph \mathcal{G}_+
\mathcal{G}_{c+}	Set of remaining edges related to $\mathcal{G}_{\tau+}$
$\mathcal{V}(\mathcal{G})$	Set of vertexes of graph \mathcal{G}
$\mathcal{E}(\mathcal{G})$	Set of edges of graph \mathcal{G}
$\mathcal{W}(\mathcal{G})$	Set of Weights of graph \mathcal{G}
$\mathcal{E}(\mathcal{G}_+) / \mathcal{E}(\mathcal{G}_-)$	Set of edges of sub-graphs $\mathcal{G}_+ / \mathcal{G}_-$
$\mathcal{V}(\mathcal{G}_+) / \mathcal{V}(\mathcal{G}_-)$	Set of vertexes of sub-graphs $\mathcal{G}_+ / \mathcal{G}_-$
$\mathcal{W}(\mathcal{G}_+) / \mathcal{W}(\mathcal{G}_-)$	Set of Weights of sub-graphs $\mathcal{G}_+ / \mathcal{G}_-$
w_{ij}	Weight value for edge (i, j)
W	Diagonal weight matrix of graph \mathcal{G}
W_+ / W_-	Diagonal weight matrices of sub-graphs $\mathcal{G}_+ / \mathcal{G}_-$
$\mathcal{C}(\mathcal{G})$	Cut-set of graph \mathcal{G}

$P(\mathcal{G}) / Q(\mathcal{G})$	Two disjoint subsets of $\mathcal{V}(\mathcal{G})$
$\mathcal{Z}(\mathcal{G})$	Effective resistance of graph \mathcal{G}
$\mathcal{I}(\mathcal{G})$	Incidence matrix of graph \mathcal{G}
$\mathcal{L}(\mathcal{G})$	Laplacian matrix of graph \mathcal{G}
$\mathcal{L}^*(\mathcal{G})$	Moore-Penrose inverse of $\mathcal{L}(\mathcal{G})$
$\mathcal{I}(\mathcal{G}_+) / \mathcal{I}(\mathcal{G}_-)$	Laplacian matrix of sub-graphs $\mathcal{G}_+ / \mathcal{G}_-$
$\mathcal{I}(\mathcal{G}_{\tau+}) \setminus \mathcal{I}(\mathcal{G}_{c+})$	Laplacian matrix of $\mathcal{G}_{\tau+} / \mathcal{G}_{c+}$
$\mathcal{I}(\mathcal{G}_k)$	Laplacian matrix of \mathcal{G}_k
$\mathcal{L}(\mathcal{G}_+) \setminus \mathcal{L}(\mathcal{G}_-)$	Laplacian matrix of sub-graphs $\mathcal{G}_+ / \mathcal{G}_-$
$\mathcal{H}(\mathcal{G})$	Connected component of sub-graph \mathcal{G}
$\mathcal{R}(\mathcal{G}_+)$	Basis for the cut space of sub-graph \mathcal{G}_+
$\mathcal{Z}(\mathcal{G}_+)$	Effective resistance of sub-graph \mathcal{G}_+
$\mathcal{T}(\mathcal{G}_{c+})$	Linear combination matrix between $\mathcal{I}(\mathcal{G}_{c+})$ and $\mathcal{I}(\mathcal{G}_{\tau+})$
$\mathcal{T}(\mathcal{G}_+)$	Linear combination matrix between $\mathcal{I}(\mathcal{G}_+)$ and $\mathcal{I}(\mathcal{G}_{\tau+})$
$\mathcal{S}(\mathcal{G}_{\tau+})$	Constructed matrix from $\mathcal{I}(\mathcal{G}_{\tau+})$
$\sigma(\cdot)$	Spectral space of a matrix
x_{uv}	Reactance of the line between u and v

I. INTRODUCTION

A. Review

Direct Current (DC) power flow has been widely utilized in high-voltage transmission network of power systems due to its simple, reliable and fast computation performance [1, 2], especially in the power market computation and analysis, including Security Constrained Economic Dispatch (SCED), Security Constrained Unit Commitment (SCUC), and Locational Marginal Price (LMP) calculation [3]. Recently, DC power flow is adopted in contingency screening, transmission switching, power system planning and reliability evaluation [4].

Generally, DC power flow is an approximation to the precise Alternating Current (AC) power flow, which ignores the reactive power and voltage magnitudes, as well as the branch resistances. This results in a completely linear power flow equations, following as

$$\mathbf{B}\Delta\theta = \Delta\mathbf{P} \quad (1)$$

where $\Delta\theta$ and $\Delta\mathbf{P}$ are derivations of voltage angles and active power injection, respectively; \mathbf{B} is a constant and sparse admittance matrix. Meanwhile, it is seen from the definition of Laplacian matrix that the original full B matrix can be treated as

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T. Ding is with the State Key Laboratory of Electrical Insulation and Power Equipment, School of Electrical Engineering, Xi'an Jiaotong University, Xi'an, Shaanxi, 710049, China. (e-mail: tding15@mail.xjtu.edu.cn)

R. Bo is with the Department of Electrical and Computer Engineering at Missouri University of Science and Technology, Rolla, MO 65409, USA

Y. Yang and F. Blaabjerg are with the Department of Energy Technology, Aalborg University, Aalborg DK-9220, Denmark.

a Laplacian matrix. In that case, the graph weight is $w_{ij}=1/x_{ij}$ for the transmission line from bus i to bus j , where x_{ij} is the reactance of the transmission line between bus i and bus j .

Throughout this paper, we consider a power network connected with n buses in the DC power flow analysis, which only has one connected component. Therefore, there is only one zero eigenvalue in the \mathbf{B} matrix. To solve the DC power flow, one bus is taken as a reference bus and the row and column of the \mathbf{B} matrix corresponding to the slack bus and the reference angle are eliminated. Obviously, the reduced $(n-1) \times (n-1)$ \mathbf{B} matrix will have a full rank that is invertible, so this linear system can be easily solved.

Traditionally, the linear equation system is solved by a direct method with sparse techniques, such as the LU decomposition [5, 6]. The process can be carried out in $\mathcal{O}(n^3)$ basic floating point operations [7]. In contrast, other approaches were based on the application of iterative methods, such as the conjugate gradient method (CG) [8, 9], minimum residual method (MINRES) [10], generalized the minimal residual method (GMRES) [11, 12], and the *Arnoldi* method [13] etc. All these methods can be categorized as the projection methods on different *Krylov subspaces*, where CG is widely used for symmetric and positive definite coefficient matrix \mathbf{B} ; MINRES is usually applicable to symmetric but indefinite coefficient matrix by introducing m -step interruptible *Lanczos process* [14]; GMRES and *Arnoldi* method are extended to asymmetric linear systems. The convergence of iterative methods depends on the *condition number* of the coefficient matrix \mathbf{B} . In order to accelerate the convergence rate, one often transforms the original linear system into another linear system with a smaller *condition number*. This process is known as *preconditioning* [15, 16] and there are many options for the *preconditioner*, such as incomplete LU factorization [17], *Chebyshev* polynomials [18, 19] and approximate inverse preconditioners [20].

B. Problems

However, it is recently observed that the “negative reactance” always exists in real-world power system models, such as the IEEE 300-bus system (one “negative reactance”) in MATPOWER [21] and the Polish power system (10-12 “negative reactances”), as well as the Northwest Power Grid in China (more than 150 “negative reactances”). As discussed in [22], the “negative reactance” is caused by the series compensation in transmission lines. It may also be resulted from the mutual effects between windings of the three-winding transformers [23].

The negative reactance may make the reduced B-matrix be singular, which in turn challenges traditional viewpoints that the reduced B-matrix is always strict positive definite [24] and [25]. As a result, DC power flow (i.e., the linear system in (1)) may be infeasible in some cases.

C. Discussions

When the reduced B matrix is singular, the original full B matrix will have more than one zero-eigenvalues and the DC power flow will be infeasible. It means that the power balance cannot be satisfied through the network. The physical meaning of this phenomenon is that the parallel/series resonance occurs due to the reciprocal of the positive and negative reactance effects.

It should be noted that [22] only pointed out the fact that the negative reactance impacts the Jacobian matrix of AC power flow and the type-1 low-voltage power-flow solutions, but how the negative reactance affected the feasibility of power flow still hasn’t been addressed.

As discussed in [26]-[28], the negative weights will affect the consensus protocol performance in the control of multi-agent systems. With the multi-agent services becoming more and more popular in power systems, this rare yet important phenomenon may be helpful for the future cyber physical system (CPS). For example, the analysis of negative reactance is beneficial to designing the network parameter to avoid the infeasibility of power flow by the negative reactance.

D. Contributions

The contributions of this paper can be summarized as:

- The definiteness of Laplacian matrix with negative weights has been explored, which shows that the positive semi-definiteness of the B-matrix should be conditionally guaranteed.
- The phenomenon of “physical dis-connectivity” may emerge at the presence of the negative reactance, which may lead the DC power flow to be infeasible. To our best knowledge, it is the first time to point out that the DC power flow may be infeasible due to the negative reactance.

The rest of the paper is organized as follows. Section II presents the general graph theory and definiteness of the Laplacian matrix. Section III shows theoretic studies related to the definiteness of Laplacian matrix with negative weights. The phenomenon of “physical dis-connectivity” resulted from negative reactance and some discussions on the feasibility of DC power flow are addressed in Section IV. In Section V, numerical results on several large-scale systems are studied to demonstrate the effectiveness of the given theorems and corollaries. Finally, conclusions are drawn in Section VI.

II. GRAPH THEORY AND DEFINITENESS OF LAPLACIAN MATRIX

A. Preliminaries and Notations for Graph

We consider an undirected edge-weighted algebraic simple graph $\mathcal{G} = \{\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}), \mathcal{W}(\mathcal{G})\}$ with vertex set $\mathcal{V}(\mathcal{G})$ and edge set $\mathcal{E}(\mathcal{G}) \subseteq \mathcal{V}(\mathcal{G}) \times \mathcal{V}(\mathcal{G})$, as well as weight value $\mathcal{W}(\mathcal{G}): \mathcal{E}(\mathcal{G}) \rightarrow \mathbb{R} \setminus \{0\}$. For each edge connecting vertices i and j , the weight w_{ij} will be assigned. The weights of all the edges will be kept in a diagonal matrix $\mathbf{W} \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{E}|}$, where $|\cdot|$ is the cardinality of a set. Let \mathbf{I} , \mathbf{J} and $\mathbf{0}$ be the unit matrix, and the matrices all of whose elements are equal to unity and zero, respectively. “ \succeq ” is the positive definiteness operator. $\mathbf{A} \succeq \mathbf{B}$ denotes that the matrix $\mathbf{A} - \mathbf{B}$ is a positive semi-definite matrix. Other related definitions are as follows [29]-[34].

- (i) A cut of a graph partitions $\mathcal{V}(\mathcal{G})$ into two disjoint subsets $P(\mathcal{G})$ and $Q(\mathcal{G})$. The cut-set $\mathcal{C}(\mathcal{G})$ is the set of edges that have one endpoint in $P(\mathcal{G})$ and the other one in $Q(\mathcal{G})$, such

that $\mathcal{C}(\mathcal{G}) = \{(u, v) \in \mathcal{E}(\mathcal{G}) \mid u \in P(\mathcal{G}), v \in Q(\mathcal{G})\}$, where $P(\mathcal{G}) \subseteq \mathcal{G}(\mathcal{G}), Q(\mathcal{G}) \subseteq \mathcal{G}, P(\mathcal{G}) \cap Q(\mathcal{G}) = \emptyset$.

(ii) Incidence matrix is defined as $\mathcal{I}(\mathcal{G}) \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{V}|}$:

$$\mathcal{I}(\mathcal{G})_{ij} = \begin{cases} -1 & \text{if } \mathcal{E}_i \text{ enters } \mathcal{V}_j \\ +1 & \text{if } \mathcal{E}_i \text{ leaves } \mathcal{V}_j \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

It is not difficult to attain $\mathcal{I}(\mathcal{G})\mathbf{J} = \mathbf{0}$.

(iii) Laplacian matrix is defined as $\mathcal{L}(\mathcal{G}) \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{V}|}$:

$$\mathcal{L}(\mathcal{G})_{ij} = \begin{cases} -w_{ij} & \text{if } i \neq j \\ \sum_{k \neq i} w_{ik} & \text{if } i = j \end{cases} \quad (3)$$

Obviously, $\mathcal{L}(\mathcal{G})$ is symmetric with real eigenvalues, which also can be defined as

$$\mathcal{L}(\mathcal{G}) = \mathcal{I}(\mathcal{G})^T \mathbf{W} \mathcal{I}(\mathcal{G}) \quad (4)$$

(iv) $\mathcal{H}(\mathcal{G})$ is called as a connected component of \mathcal{G} , if $\mathcal{H}(\mathcal{G})$ is a connected sub-graph of \mathcal{G} , and is not contained in any other connected sub-graph of \mathcal{G} that has more vertices or edges than \mathcal{H} has. Fig.1 gives exemplifies a graph with four components. Besides, a connected graph has only one component.

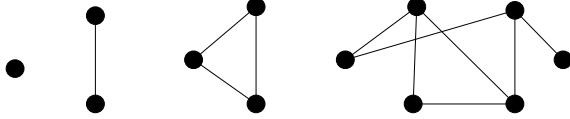


Fig. 1 A graph with four connected components

(v) A graph is called a connected graph if there is a path from any point to any other point in the graph. There is only one component for a connected graph. If a graph is not connected, it has more than one component.

(vi) For the graph \mathcal{G} , we define the sub-graph \mathcal{G}_+ and \mathcal{G}_- to be the sub-graphs with the same vertex set as \mathcal{G} , together with the positive and negative edge weights, respectively, such that

$$\mathcal{G}_+ = \{\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}_+), \mathcal{W}(\mathcal{G}_+)\},$$

$$\text{where} \begin{cases} \mathcal{W}(\mathcal{G}_+) = \max(\mathcal{W}(\mathcal{G}), 0) \\ \mathcal{E}(\mathcal{G}_+) \subseteq \mathcal{E}(\mathcal{G}), \text{ with } \mathcal{W}(\mathcal{G}) \in \mathcal{W}(\mathcal{G}_+) \end{cases} \quad (5)$$

$$\mathcal{G}_- = \{\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}_-), \mathcal{W}(\mathcal{G}_-)\},$$

$$\text{where} \begin{cases} \mathcal{W}(\mathcal{G}_-) = \min(\mathcal{W}(\mathcal{G}), 0) \\ \mathcal{E}(\mathcal{G}_-) \subseteq \mathcal{E}(\mathcal{G}), \text{ with } \mathcal{W}(\mathcal{G}) \in \mathcal{W}(\mathcal{G}_-) \end{cases} \quad (6)$$

It is obvious that $[\mathcal{E}(\mathcal{G}_+)] + [\mathcal{E}(\mathcal{G}_-)] = [\mathcal{E}(\mathcal{G})]$ and $[\mathcal{V}(\mathcal{G}_+)] = [\mathcal{V}(\mathcal{G}_-)] = [\mathcal{V}(\mathcal{G})]$.

B. Definiteness of Laplacian Matrix

In order to study the definiteness of Laplacian matrix, we will give some properties related to the eigenvalues. It has been reported in [27] that the Laplacian matrix defined by (3) has some properties that

(i) Laplacian is symmetric and diagonally dominant, so its eigenvalues are nonnegative according to *Gershgorin* theory.

(ii) Note that $\mathcal{L}(\mathcal{G})\mathbf{J} = \mathbf{0}$. It shows that there is at least one zero-eigenvalue in $\mathcal{L}(\mathcal{G})$.

(iii) The number of 0 appears as an eigenvalue in the Laplacian is the number of connected components in the graph.

Moreover, for a connected graph with all positive weights, the Laplacian matrix is semi-definite, and it has only one zero-eigenvalue and all the others are strictly positive.

Especially, for a connected graph $\mathcal{L}(\mathcal{G})$ with all positive weights, let the eigenvalues be $\lambda_1 = 0 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots \leq \lambda_{|\mathcal{V}|}$. Since there is one zero-eigenvalue, the determinant of $\mathcal{L}(\mathcal{G})$ is null and hence $\mathcal{L}(\mathcal{G})$ cannot be inverted. We can use the *Moore-Penrose* generalized inverse of $\mathcal{L}(\mathcal{G})$ to substitute the inverse of $\mathcal{L}(\mathcal{G})$, denoted as $\mathcal{L}^\dagger(\mathcal{G})$. To compute $\mathcal{L}^\dagger(\mathcal{G})$, it admits the spectral decomposition of $\mathcal{L}(\mathcal{G})$ to be

$$\mathcal{L}(\mathcal{G}) = \mathbf{U}^T \mathbf{\Lambda} \mathbf{U} \quad (7)$$

where $\mathbf{\Lambda} = \text{diag}(\lambda_1, \dots, \lambda_{|\mathcal{V}|})$ and \mathbf{U} is an orthogonal matrix whose columns correspond the eigenvectors of $\mathcal{L}(\mathcal{G})$ and $\mathbf{U}^T = \mathbf{U}^{-1}$. According to this spectral decomposition, the inverse of $\mathcal{L}(\mathcal{G})$ is defined as

$$\mathcal{L}^\dagger(\mathcal{G}) = \mathbf{U}^{-1} \mathbf{\Lambda}^{-1} (\mathbf{U}^T)^{-1} = \mathbf{U}^T \mathbf{\Sigma} \mathbf{U} \quad (8)$$

where $\mathbf{\Sigma} = \text{diag}(\alpha_1, \dots, \alpha_{|\mathcal{V}|})$ with $\alpha_i = \begin{cases} 1/\lambda_i & i \geq 2 \\ 0 & i = 1 \end{cases}$.

With respect to the Moore-Penrose pseudo-inverse of Laplacian matrix, we will introduce the definition of effective resistance between any two vertices $u, v \in \mathcal{V}$ in graph \mathcal{G} , denoted by $\mathcal{Z}_{uv}(\mathcal{G})$, is

$$\mathcal{Z}_{uv}(\mathcal{G}) = (\mathbf{e}_u - \mathbf{e}_v)^T \mathcal{L}^\dagger(\mathcal{G}) (\mathbf{e}_u - \mathbf{e}_v) \\ = \mathcal{L}_{uu}^\dagger(\mathcal{G}) + \mathcal{L}_{vv}^\dagger(\mathcal{G}) - \mathcal{L}_{uv}^\dagger(\mathcal{G}) - \mathcal{L}_{vu}^\dagger(\mathcal{G}) \quad (9)$$

where \mathbf{e}_u is a unit vector in which the element in the u position is 1, and 0 elsewhere. The effective resistance reflects the electrical distance between two distinct vertices, which also satisfies the Ohm's and Kirchhoff's laws [28].

For this connected graph \mathcal{G} , it can be observed from (8) that

$$\mathcal{L}_{uv}^\dagger(\mathcal{G}) = \sum_{k=1}^{|\mathcal{V}|} \alpha_k U_{uk} U_{vj} \quad (10)$$

where $\alpha_k \geq 0$, $k = 1, \dots, |\mathcal{V}|$. The effective resistance between u and v by (9) can be expressed as

$$\mathcal{Z}_{uv}(\mathcal{G}) = \mathcal{L}_{uu}^\dagger(\mathcal{G}) + \mathcal{L}_{vv}^\dagger(\mathcal{G}) - \mathcal{L}_{uv}^\dagger(\mathcal{G}) - \mathcal{L}_{vu}^\dagger(\mathcal{G}) \\ = \sum_{k=1}^{|\mathcal{V}|} \alpha_k U_{uk} U_{ku} - 2 \sum_{k=1}^{|\mathcal{V}|} \alpha_k U_{uk} U_{vj} + \sum_{k=1}^{|\mathcal{V}|} \alpha_k U_{vk} U_{kv} \quad (11) \\ = \sum_{k=1}^{|\mathcal{V}|} \alpha_k (U_{uk} - U_{vk})^2 > 0$$

In addition, we will provide a theorem for a symmetric matrix, which will be further used for the Laplacian matrix.

Theorem 1 (Courant-Fisher min-max principle) [29]: The equality holds for the k -th λ_k eigenvalue of a symmetric matrix M , where $\lambda_1 \leq \dots \leq \lambda_k \dots \leq \lambda_{|V|}$.

$$\lambda_k = \min_{S_k} \max_{x \neq 0, x \in S_k} \frac{\langle Mx, x \rangle}{\langle x, x \rangle} \quad (12)$$

where S_k is a k -dimensional subset of $\mathbb{R}^{|V|}$.

III. IMPACT OF NEGATIVE WEIGHTS ON THE DEFINITENESS OF LAPLACIAN MATRIX

It should be noted that the negative weights will lead to the loss of the diagonally dominant property, which will further affect the definiteness of the Laplacian matrix. In this subsection, we will discuss that the definiteness of the Laplacian is mainly related to the location and magnitude of the negative weight edges. At first, we will show the following corollary.

Corollary 1: Let \mathcal{G} be a graph and $\bar{\mathcal{G}}$ be obtained by increasing the weight of an edge, and the eigenvalues of the new graph satisfy

$$\lambda_k(\bar{\mathcal{G}}) \geq \lambda_k(\mathcal{G}) \quad (13)$$

Proof: Supposing that the weight w_{ij} of the edge $l=(i, j)$ is increased by an amount $\varepsilon > 0$, construct Γ to be a $[\mathcal{E}] \times [\mathcal{E}]$ diagonal matrix with only one non-zero element ε at (l, l) , so $\Gamma \succeq 0$. It then yields

$$\mathcal{L}(\bar{\mathcal{G}}) = \mathcal{I}(\mathcal{G})^T (W + \Gamma) \mathcal{I}(\mathcal{G}) = \mathcal{L}(\mathcal{G}) + \mathcal{I}(\mathcal{G})^T \Gamma \mathcal{I}(\mathcal{G}) \succeq \mathcal{L}(\mathcal{G}) \quad (14)$$

According to **Theorem 1**, the k -th eigenvalue of the new graph can be formulated as

$$\begin{aligned} \lambda_k(\bar{\mathcal{G}}) &= \min_{S_k} \max_{x \neq 0, x \in S_k} \frac{\langle \mathcal{L}(\bar{\mathcal{G}})x, x \rangle}{\langle x, x \rangle} \\ &= \min_{S_k} \max_{x \neq 0, x \in S_k} \frac{\langle (\mathcal{L}(\mathcal{G}) + \mathcal{I}(\mathcal{G})^T \Gamma \mathcal{I}(\mathcal{G}))x, x \rangle}{\langle x, x \rangle} \\ &\geq \min_{S_k} \max_{x \neq 0, x \in S_k} \frac{\langle \mathcal{L}(\mathcal{G})x, x \rangle}{\langle x, x \rangle} = \lambda_k(\mathcal{G}) \end{aligned} \quad (15)$$

Then, **Corollary 1** is proved.

(Q.E.D.)

Corollary 1 concludes that the decrease of the weight will not increase the value of eigenvalues. Moreover, adding an edge with the positive weight will not decrease the value of eigenvalues. Similarly, adding an edge with the negative weight will not increase the value of eigenvalues. This implies that a larger negative weight has a larger possibility to result in the loss of the positive definiteness of Laplacian matrix and when the value of negative weight is small, the positive definiteness couldn't be change. However, how to select the negative weight to guarantee the positive definiteness of the Laplacian matrix should be investigated. The following corollary will provide a sufficient and necessary condition.

Corollary 2: (Sufficient and necessary condition) If \mathcal{G}_+ is a connected graph, Laplacian matrix of the graph with the presence of negative weights could be positive semi-definite if and only if $|W_-|^{-1} - \mathcal{I}(\mathcal{G}_-) \mathcal{L}^*(\mathcal{G}_+) \mathcal{I}(\mathcal{G}_-)^T \succeq 0$.

Proof: If we partition the incidence matrix into two separate parts with positive and negative weights respectively as

$$\mathcal{I}(\mathcal{G}) = \begin{bmatrix} \mathcal{I}(\mathcal{G}_+) \\ \mathcal{I}(\mathcal{G}_-) \end{bmatrix}, \text{ the Laplacian matrix can be reformulated as}$$

$$\begin{aligned} \mathcal{L}(\mathcal{G}) &= \mathcal{I}(\mathcal{G})^T W \mathcal{I}(\mathcal{G}) = \begin{bmatrix} \mathcal{I}(\mathcal{G}_+) \\ \mathcal{I}(\mathcal{G}_-) \end{bmatrix}^T \begin{bmatrix} W_+ & 0 \\ 0 & W_- \end{bmatrix} \begin{bmatrix} \mathcal{I}(\mathcal{G}_+) \\ \mathcal{I}(\mathcal{G}_-) \end{bmatrix} \\ &= \mathcal{I}(\mathcal{G}_+)^T W_+ \mathcal{I}(\mathcal{G}_+) + \mathcal{I}(\mathcal{G}_-)^T W_- \mathcal{I}(\mathcal{G}_-) \\ &= \mathcal{I}(\mathcal{G}_+)^T W_+ \mathcal{I}(\mathcal{G}_+) - \mathcal{I}(\mathcal{G}_-)^T |W_-| \mathcal{I}(\mathcal{G}_-) \\ &= \mathcal{L}(\mathcal{G}_+) - \mathcal{L}(\mathcal{G}_-) \end{aligned} \quad (16)$$

By the *Schur complement* [35], $\mathcal{L}(\mathcal{G}) \succeq 0$ if and only if

$$\begin{bmatrix} |W_-|^{-1} & \mathcal{I}(\mathcal{G}_-) \\ \mathcal{I}(\mathcal{G}_-)^T & \mathcal{L}(\mathcal{G}_+) \end{bmatrix} \succeq 0 \quad (17)$$

Since $\mathcal{L}(\mathcal{G}_+)$ is a connected graph, $\mathcal{L}^*(\mathcal{G}_+)$ can be obtained from (8). Meanwhile, $|W_-|^{-1} \succeq 0$ and thus applying the *Schur complement* again to (17), the equivalent condition is obtained as

$$|W_-|^{-1} - \mathcal{I}(\mathcal{G}_-) \mathcal{L}^*(\mathcal{G}_+) \mathcal{I}(\mathcal{G}_-)^T \succeq 0 \quad (18)$$

(Q.E.D.)

Furthermore, (18) leads to a desired conclusion that $|W_-| \preceq (\mathcal{I}(\mathcal{G}_-) \mathcal{L}^*(\mathcal{G}_+) \mathcal{I}(\mathcal{G}_-)^T)^{-1}$. However, the inverse of $\mathcal{I}(\mathcal{G}_-) \mathcal{L}^*(\mathcal{G}_+) \mathcal{I}(\mathcal{G}_-)^T$ should conditionally exist. In the following work, we will give a condition to guarantee the inverse of $\mathcal{I}(\mathcal{G}_-) \mathcal{L}^*(\mathcal{G}_+) \mathcal{I}(\mathcal{G}_-)^T$.

Corollary 3: Under the assumption that \mathcal{G}_+ is a connected graph and there is no cycle in \mathcal{G}_- , the inverse of $\mathcal{I}(\mathcal{G}_-) \mathcal{L}^*(\mathcal{G}_+) \mathcal{I}(\mathcal{G}_-)^T$ exists. The Laplacian matrix of the graph with negative weights can keep definite if and only if $|W_{uv}| \leq \mathcal{Z}_{uv}(\mathcal{G})^{-1}$ for all $(u, v) \in \mathcal{E}_-$.

Proof: Since \mathcal{G}_+ is a connected graph, it can be written as the union of two edge-disjoint sub-graphs on the same vertex set as $\mathcal{G}_+ = \mathcal{G}_{\tau+} \cup \mathcal{G}_{c+}$, where $\mathcal{G}_{\tau+}$ is a spanning tree and \mathcal{G}_{c+} contains the remaining edges that necessarily complete the cycles in \mathcal{G}_+ . The rows of the incidence matrix for the graph \mathcal{G}_+ can always be permuted such that

$$\mathcal{I}(\mathcal{G}_+) = \begin{bmatrix} \mathcal{I}(\mathcal{G}_{\tau+}) \\ \mathcal{I}(\mathcal{G}_{c+}) \end{bmatrix} \quad (19)$$

The cycle edges can be represented as the linear combinations of the tree edges according to [36].

$$\mathcal{I}(\mathcal{G}_{c+}) = \mathcal{I}(\mathcal{G}_{c+}) \mathcal{I}(\mathcal{G}_{\tau+}) \quad (20)$$

where

$$\mathcal{I}(\mathcal{G}_{c+}) = \mathcal{I}(\mathcal{G}_{c+})^T \mathcal{I}(\mathcal{G}_{\tau+}) (\mathcal{I}(\mathcal{G}_{\tau+})^T \mathcal{I}(\mathcal{G}_{\tau+}))^{-1} \quad (21)$$

Alternatively, the representation of the incidence matrix can be reformulated as

$$\mathcal{I}(\mathcal{G}_+) = \mathcal{R}(\mathcal{G}_+) \mathcal{I}(\mathcal{G}_{\tau_+}), \quad \mathcal{R}(\mathcal{G}_+) = \begin{bmatrix} \mathbf{I} \\ \mathcal{T}(\mathcal{G}_+) \end{bmatrix} \quad (22)$$

In fact, the matrix $\mathcal{R}(\mathcal{G}_+)$ can be viewed as the basis for the cut space of \mathcal{G}_+ . Applying (23) to the Laplacian matrix (4), we have

$$\begin{cases} \mathcal{S}(\mathcal{G}_{\tau_+}) = \begin{bmatrix} \mathcal{I}(\mathcal{G}_{\tau_+})^T (\mathcal{I}(\mathcal{G}_{\tau_+}) \mathcal{I}(\mathcal{G}_{\tau_+})^T)^{-1} & \mathbf{J} \\ \mathcal{S}(\mathcal{G}_{\tau_+})^{-1} = \begin{bmatrix} \mathcal{I}(\mathcal{G}_{\tau_+}) \\ \frac{1}{|\mathcal{V}|} \mathbf{J}^T \end{bmatrix} \end{cases} \quad (23)$$

$$\begin{aligned} \mathcal{S}(\mathcal{G}_{\tau_+})^{-1} \mathcal{L}(\mathcal{G}_+) \mathcal{S}(\mathcal{G}_{\tau_+}) \\ = \begin{bmatrix} \mathcal{I}(\mathcal{G}_{\tau_+}) \mathcal{I}(\mathcal{G}_{\tau_+})^T \mathcal{R}(\mathcal{G}_+)^T \mathbf{W}_+ \mathcal{R}(\mathcal{G}_+) & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} \end{aligned} \quad (24)$$

Equation (24) provides a transparent way to isolate zero eigenvalues of the Laplacian for the graph \mathcal{G}_+ . It should be noted that rows of $\mathcal{I}(\mathcal{G}_{\tau_+})$ are linearly independent, so the inverse matrix $\mathcal{I}(\mathcal{G}_{\tau_+}) \mathcal{I}(\mathcal{G}_{\tau_+})^T$ exists and has a full rank. It can be found that the eigenvalues of $\mathcal{R}(\mathcal{G}_+)^T \mathbf{W}_+ \mathcal{R}(\mathcal{G}_+)$ correspond to the non-zero eigenvalues of $\mathcal{L}(\mathcal{G}_+)$, such that

$$\sigma(\mathcal{R}(\mathcal{G}_+)^T \mathbf{W}_+ \mathcal{R}(\mathcal{G}_+)) = \sigma_+(\mathcal{L}(\mathcal{G}_+)) \quad (25)$$

where $\sigma(\cdot)$ denotes the spectral space of a matrix. Take the inverse for (24) as

$$\begin{aligned} \mathcal{S}(\mathcal{G}_{\tau_+})^{-1} \mathcal{L}^*(\mathcal{G}_+) \mathcal{S}(\mathcal{G}_{\tau_+}) \\ = \begin{bmatrix} (\mathcal{R}(\mathcal{G}_+)^T \mathbf{W}_+ \mathcal{R}(\mathcal{G}_+))^{-1} (\mathcal{I}(\mathcal{G}_{\tau_+}) \mathcal{I}(\mathcal{G}_{\tau_+})^T)^{-1} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} \end{aligned} \quad (26)$$

Equation (26) is equal to

$$\mathcal{L}^*(\mathcal{G}_+) = \mathcal{S}(\mathcal{G}_{\tau_+}) \begin{bmatrix} (\mathcal{R}(\mathcal{G}_+)^T \mathbf{W}_+ \mathcal{R}(\mathcal{G}_+))^{-1} (\mathcal{I}(\mathcal{G}_{\tau_+}) \mathcal{I}(\mathcal{G}_{\tau_+})^T)^{-1} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} \mathcal{S}(\mathcal{G}_{\tau_+})^{-1} \quad (27)$$

Substituting $\mathcal{S}(\mathcal{G}_{\tau_+})$ and $\mathcal{S}(\mathcal{G}_{\tau_+})^{-1}$ into (27) yields (28).

According to the condition that there is no cycle in \mathcal{G}_- , \mathcal{G}_- will have a forest structure. Thus, we find a spanning tree \mathcal{G}_k in \mathcal{G} which contains \mathcal{G}_- . Therefore, we define the transformation as shown in (29).

Due to $\mathcal{G}_+ \subseteq \mathcal{G}$ with the same vertex, it leads to $\mathcal{G}_{\tau_+} \subseteq \mathcal{G}$. In this term, \mathcal{G}_k and \mathcal{G}_{τ_+} are two different spanning trees of \mathcal{G} , so $\mathcal{I}(\mathcal{G}_{\tau_+}) \mathcal{I}(\mathcal{G}_k)^T$ must have a full rank and be reversible [37]. Then, we have

$$\mathcal{L}^*(\mathcal{G}_+) = \mathcal{I}(\mathcal{G}_{\tau_+})^T (\mathcal{I}(\mathcal{G}_{\tau_+}) \mathcal{I}(\mathcal{G}_{\tau_+})^T)^{-1} (\mathcal{R}(\mathcal{G}_+)^T \mathbf{W}_+ \mathcal{R}(\mathcal{G}_+))^{-1} (\mathcal{I}(\mathcal{G}_{\tau_+}) \mathcal{I}(\mathcal{G}_{\tau_+})^T)^{-1} \mathcal{I}(\mathcal{G}_{\tau_+}) \quad (28)$$

$$\mathcal{I}(\mathcal{G}_k) \mathcal{L}^*(\mathcal{G}_+) \mathcal{I}(\mathcal{G}_k)^T = \mathcal{I}(\mathcal{G}_k) \mathcal{I}(\mathcal{G}_{\tau_+})^T (\mathcal{I}(\mathcal{G}_{\tau_+}) \mathcal{I}(\mathcal{G}_{\tau_+})^T)^{-1} (\mathcal{R}(\mathcal{G}_+)^T \mathbf{W}_+ \mathcal{R}(\mathcal{G}_+))^{-1} (\mathcal{I}(\mathcal{G}_{\tau_+}) \mathcal{I}(\mathcal{G}_{\tau_+})^T)^{-1} \mathcal{I}(\mathcal{G}_{\tau_+}) \mathcal{I}(\mathcal{G}_k)^T \quad (29)$$

$$\begin{aligned} \sigma(\mathcal{I}(\mathcal{G}_k) \mathcal{L}^*(\mathcal{G}_+) \mathcal{I}(\mathcal{G}_k)^T) \\ = \sigma\left(\left(\mathcal{R}(\mathcal{G}_{\tau_+})^T \mathbf{W}_+ \mathcal{R}(\mathcal{G}_{\tau_+})\right)^{-1}\right) = \sigma_+(\mathcal{L}^*(\mathcal{G}_+)) \end{aligned} \quad (30)$$

Since \mathcal{G}_- is the subset of \mathcal{G}_k , we have $\mathcal{G}_- \subseteq \mathcal{G}_k$ and

$$\sigma(\mathcal{I}(\mathcal{G}_-) \mathcal{L}^*(\mathcal{G}_+) \mathcal{I}(\mathcal{G}_-)^T) \subseteq \sigma_+(\mathcal{L}^*(\mathcal{G}_+)) \quad (31)$$

This illustrates that the inverse matrix of $\mathcal{I}(\mathcal{G}_-) \mathcal{L}^*(\mathcal{G}_+) \mathcal{I}(\mathcal{G}_-)^T$ exists.

Moreover, due to the condition that there is no cycle in \mathcal{G}_- , \mathcal{G}_- will have a forest structure, so it can be verified that the matrix $\mathcal{I}(\mathcal{G}_-) \mathcal{L}^*(\mathcal{G}_+) \mathcal{I}(\mathcal{G}_-)^T$ is a $|\mathcal{G}_-| \times |\mathcal{G}_-|$ diagonal matrix. Each diagonal element corresponds to an edge with a negative weight. For the k -th individual edge (u, v) of the graph with the negative weight $w_{uv} < 0$, we have

$$\begin{aligned} (\mathcal{I}(\mathcal{G}_-) \mathcal{L}^*(\mathcal{G}_+) \mathcal{I}(\mathcal{G}_-)^T)_{kk} = \\ [0, \dots, 1, 0, \dots, -1, 0, \dots, 0] \mathcal{L}^*(\mathcal{G}_+) [0, \dots, 1, 0, \dots, -1, 0, \dots, 0]^T \end{aligned} \quad (32)$$

From the definition of effective resistance in (16), we can find that (32) is just the effective resistance between u and v in the graph \mathcal{G}_+ . Thus, we can obtain

$$\mathcal{I}(\mathcal{G}_-) \mathcal{L}^*(\mathcal{G}_+) \mathcal{I}(\mathcal{G}_-)^T = \text{diag}(\mathcal{Z}_1(\mathcal{G}_+), \mathcal{Z}_2(\mathcal{G}_+), \dots, \mathcal{Z}_{|\mathcal{G}_-|}(\mathcal{G}_+)) \quad (33)$$

According to the sufficient and necessary condition, it gives the fact that the Laplacian matrix with negative weights could keep definite if and only if $|\mathcal{W}_{uv}| \leq \mathcal{Z}_{uv}(\mathcal{G})^{-1}$ for all $(u, v) \in \mathcal{E}_-$.

(Q.E.D.)

This corollary indicates that if the absolute value of the negative weight is smaller than its inverse of the positive effective resistance, the Laplacian matrix will be positive semi-definite. It is desired to note that the definition of the effective resistance in (9) is for the graph with only positive weights. For the graph with the presence of negative weights, we can extend this concept by splitting the positive and negative parts of the graph and then combining them together. Fig. 2 gives the intuitive physical interpretation that the general equivalent resistance between u and v in the graph \mathcal{G} can be regarded as two parallel resistances $\mathcal{Z}_{uv}(\mathcal{G}_+)$ and w_{uv}^{-1} , such that

$$\mathcal{C}_{uv}(\mathcal{G}) = \frac{w_{uv}^{-1} \mathcal{Z}_{uv}(\mathcal{G}_+)}{\mathcal{Z}_{uv}(\mathcal{G}_+) + w_{uv}^{-1}} \quad (34)$$

Obviously, if $|w_{uv}| < \mathcal{Z}_{uv}^{-1}(\mathcal{G}_+)$ (or $|w_{uv}| > \mathcal{Z}_{uv}^{-1}(\mathcal{G}_+)$), the general equivalent resistance is positive, i.e., $\mathcal{C}_{uv}(\mathcal{G}) > 0$, so we will have a positive semi-definite Laplacian matrix according to **Corollary 3**. However, **Corollary 3** is based on the assumption that \mathcal{G}_+ is a connected graph.

Now we will relax this assumption. For the connected graph $\mathcal{G} = \mathcal{G}_+ \cup \mathcal{G}_-$, if its sub-graph \mathcal{G}_+ is not connected, there must be some edges of \mathcal{G}_- belonging to the cut-set of \mathcal{G} .

If the edge $\mathcal{E}_k = (u, v)$ with a negative weight belongs to the cut-set, $\mathcal{Z}_{uv}(\mathcal{G}_+) = +\infty$ and we will arrive at

$$|w_{uv}| > (\mathcal{Z}_{uv}(\mathcal{G}_+))^{-1} = 0 \quad (35)$$

We can find that **Corollary 3** cannot be satisfied for any given value of the negative weight, so **Laplacian matrix of the graph with negative weights is always indefinite if its sub-graph \mathcal{G}_+ is not connected.**

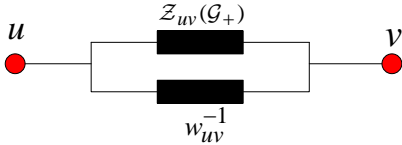


Fig. 2 Physical explanation of corollary 4 with one negative weight

IV. THE PHENOMENON OF “PHYSICAL DIS-CONNECTIVITY” AND DISCUSSIONS ON DC POWER FLOW

A. The Phenomenon of “Physical Dis-connectivity”

As discussed in the last section, the definiteness of the Laplacian matrix is related to the value and location of negative edges. It is considered in Fig. 2 that the source is connected to the node u and the sink is connected to the node v , and thus the power will be transferred from u to v . When the negative reactive equals to the inverse of positive equivalent resistance, i.e., $|w_{uv}| = \mathcal{Z}_{uv}^{-1}(\mathcal{G}_+)$, it yields $\mathcal{Z}_{uv}(\mathcal{G}_+) + w_{uv}^{-1} = 0$ and the general equivalent resistance between u and v will be infinite. It means that the graph has a cut between the vertices u and v , so that the flow cannot be transferred from u to v and the power flow is infeasible. In this sense, the graph is thought to be disconnected and split into two components.

However, such dis-connectivity is different from the traditional graph dis-connectivity based on the geometry and topology. We call the components arisen from topological isolation as “geometric dis-connectivity” (a.k.a., graph dis-connectivity). For those caused by infinite effective resistance, we call them as “physical dis-connectivity”. In fact, “geometric dis-connectivity” must lead to “physical dis-connectivity”, but “physical dis-connectivity” does not always lead to “geometric dis-connectivity”.

For the graph with all positive weights, the concept of geometric dis-connectivity and physical dis-connectivity are identical. Thus, “physical dis-connectivity” will result in “geometric dis-connectivity”. But this is not true for the graph that contains negative weights.

As discussed in the previous section, there is always one zero-eigenvalue in Laplacian matrix for a connected graph, and we let $\lambda_1 = 0$. But if there exists any negative weight in a

connected graph, we can’t guarantee the other eigenvalues λ_k ($k=2, \dots, [\mathcal{V}]$) to be non-zero. If there exists zero eigenvalues, $\alpha_k = 1/\lambda_k$ is infinite and the general equivalent resistance will become infinite. Then, physical dis-connectivity” will come up. However, from the perspective of topology, the graph is still connected, so the graph has no “geometric dis-connectivity”. Actually, the physical connectivity is closely related to the graph weights, while the geometric dis-connectivity is not.

Therefore, for a connected graph with the presence of negative weights, if there are more than one zero-eigenvalues, “geometric dis-connectivity” exists. On the other hand, for an unconnected graph, the effective resistance between different connected components is infinite which leads to “physical dis-connectivity”.

According to the above analysis, for the graph with the presence of negative weights, we obtain that “geometric dis-connectivity” must lead to “physical dis-connectivity”, but if there are “physical dis-connectivity”, “geometric dis-connectivity” won’t necessarily come up.

B. Discussion on the B Matrix and DC Power Flow

As discussed in Section II, the **B** matrix for a connected power network with positive weights has only one zero eigenvalue and the reduced **B** matrix is always invertible. However, Corollary 3 shows that the **B** matrix of the power network with the presence of the negative reactance could have more than one zero-eigenvalues and the reduced **B** matrix will be singular if $|W_-|^{-1} - \mathcal{I}(\mathcal{G}_-)\mathcal{L}^+(\mathcal{G}_+)\mathcal{I}(\mathcal{G}_-)^T$ is singular when \mathcal{G}_+ is a connected graph. This gives an exact analytic expression among multiple negative reactances that can make the DC power flow infeasible.

However, when \mathcal{G}_+ is not a connected graph, for each negative reactance, e.g., $x_{uv} < 0$, we let $\overline{\mathcal{G}_{uv-}}$ be a new graph eliminating the negative reactance (u, v) . Then, we can compute the equivalent resistance between (u, v) as

$$\mathcal{C}_{uv}(\mathcal{G}) = \frac{x_{uv}\mathcal{Z}_{uv}(\overline{\mathcal{G}_{uv-}})}{x_{uv} + \mathcal{Z}_{uv}(\overline{\mathcal{G}_{uv-}})}, \quad \forall (u, v) \in \mathcal{E}_- \quad (36)$$

In particular, if (u, v) is the cut of \mathcal{G} , $\mathcal{Z}_{uv}(\overline{\mathcal{G}_{uv-}}) = \infty$ and we have the equivalent resistance as (37). This implies that DC power flow can be feasible for any value of the negative reactance because the equivalent resistance couldn’t be infinite for any finite value of x_{uv} .

$$\mathcal{C}_{uv}(\mathcal{G}) = \lim_{\mathcal{Z}_{uv}(\overline{\mathcal{G}_{uv-}}) \rightarrow +\infty} \frac{x_{uv}\mathcal{Z}_{uv}(\overline{\mathcal{G}_{uv-}})}{\mathcal{Z}_{uv}(\overline{\mathcal{G}_{uv-}}) + x_{uv}} = x_{uv} \neq 0 \quad (37)$$

Otherwise, $\mathcal{Z}_{uv}(\overline{\mathcal{G}_{uv-}})$ is finite. At this time, if $x_{uv} + \mathcal{Z}_{uv}(\overline{\mathcal{G}_{uv-}}) = 0$, $\mathcal{C}_{uv}(\mathcal{G})$ will be infinite and thus DC power flow will be infeasible. Note that the number of the negative reactances in $\overline{\mathcal{G}_{uv-}}$ is one less than that in \mathcal{G} . This expression gives an implicit relationship between one negative reactance and the other negative reactances. The relationship describes a new phenomenon that DC power flow will be infeasible.

Taking one negative reactance for illustration, we have two cases for the k -th line with the negative reactance which can be shown in Fig. 3. With increasing the weight, the eigenvalue will increase with respect to **Corollary 1**.

(i) If \mathcal{G}_+ is an unconnected graph, the k -th edge belongs to the cut-set of different components in \mathcal{G}_+ and $\mathcal{Z}_k(\mathcal{G}_+) = +\infty$. For the positive weight x_k , all eigenvalues are non-negative. For the negative weight x_k , we have the general equivalent resistance as $\mathcal{C}_k(\mathcal{G}) = \lim_{\mathcal{Z}_k(\mathcal{G}_+) \rightarrow +\infty} \frac{x_k \mathcal{Z}_k(\mathcal{G}_+)}{\mathcal{Z}_k(\mathcal{G}_+) + x_k} = x_k$. Therefore, $\mathcal{C}_k(\mathcal{G})$ cannot be infinite and there is no “physical dis-connectivity” unless x_k is close to infinite value.

(ii) If \mathcal{G}_+ is a connected graph, $\mathcal{Z}_k(\mathcal{G}_+) > 0$. For the positive weight x_k , all eigenvalues are non-negative, while for the negative weight x_k , the sign of the undetermined eigenvalue can be obtained by **Corollary 2**. At this moment, there may exist “physical dis-connectivity” when $|x_k|$ equals to $\mathcal{Z}_k(\mathcal{G}_+)$.

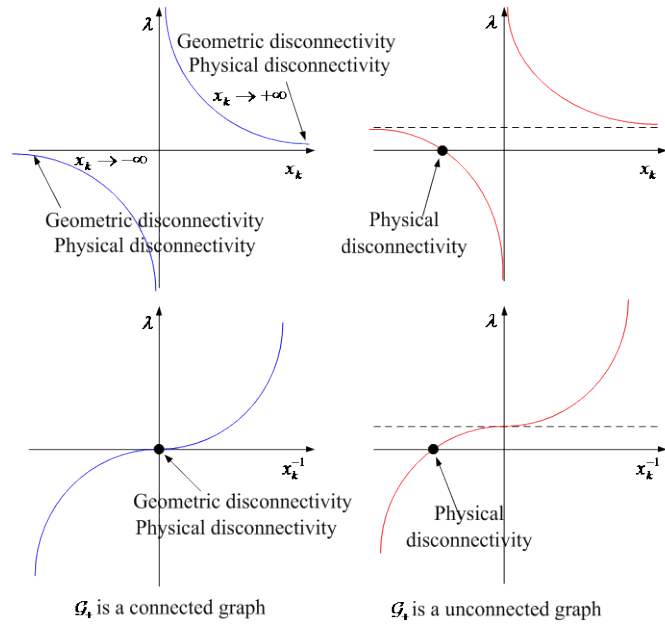


Fig. 3 Comparison of “physical dis-connectivity” and “geometric dis-connectivity”

From the above analysis, we can find the following observation: The value of negative reactance will affect the eigenvalues of the \mathbf{B} matrix, which will further lead to a new phenomenon that the DC power flow for a connected power system may be infeasible, while this phenomenon doesn't occur in the traditional situation where the value of the reactance for all the transmission lines is positive. Specifically, if the \mathbf{B} matrix has more than one zero-eigenvalues, the DC power flow will be infeasible. This phenomenon is called as “physical dis-connectivity”.

V. NUMERICAL EXAMPLE

We investigate several large-scale test systems available from MATPOWER [21] and find that there is one negative reactance (NR) in IEEE 300-bus case, 10 NRs in the 3012-bus and the 3120-bus real-life Polish systems and 12 NRs in the 3375-bus

Polish system. The impact of NRs on the \mathbf{B} matrix and DC power flow is studied.

A. A 4-bus Test System

To show the impact of NR on the eigenvalues of the \mathbf{B} -matrix and DC power flow, a simple 4-bus test system is designed, the topology of which is depicted in Fig. 4. The bus 1 is the reference and three loads are connected to the other buses. The load value is 10MW for bus 1, 20 MW for bus 2 and 30 MW for bus 3. At first, we consider there is one negative reactance on the line 2 and the other three lines are all with positive reactance, i.e., 0.1 p.u. for line 1, 0.4 p.u. for line 3 and 0.2 p.u. for line 4.

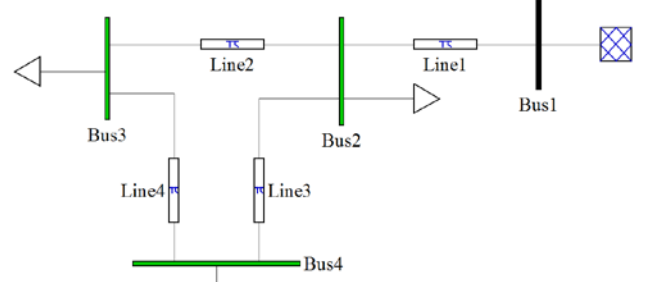


Fig. 4 4-bus test system

According to **Corollary 3**, we can find that the sub-graph \mathcal{G}_+ is a connected graph, as shown in Fig. 5. Correspondingly, the \mathbf{B} matrix of \mathcal{G}_+ can be formulated as $\mathcal{L}(\mathcal{G}_+)$ in Fig. 5 and the definition of the effective resistance in (16) is written as

$$\mathcal{I}(\mathcal{G}_+) \mathcal{L}^\dagger(\mathcal{G}_+) \mathcal{I}(\mathcal{G}_+)^T = (0, 1, -1, 0) \mathcal{L}^\dagger(\mathcal{G}_+) (0, 1, -1, 0)^T = 0.6$$

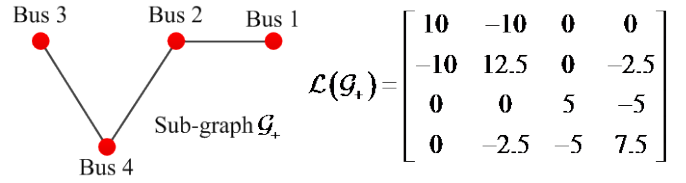


Fig. 5 4-bus test system

Then, we can conclude by **Corollary 3** that

- (i) When the value of the negative reactance belongs to $(-\infty, -0.6)$, the \mathbf{B} matrix is positive semi-definite;
- (ii) When the value of negative reactance is -0.6 , the \mathbf{B} matrix becomes singular;
- (iii) When the value of the negative reactance is within $(-0.6, 0)$, the \mathbf{B} matrix is indefinite.

Furthermore, we consider four cases under different value of the reactance on line 2. The results in Table 1 reflect the same conclusions as **Corollary 3**. When the value of the reactance is positive, the power flow can be feasible and all the eigenvalues are nonnegative; when the value of the reactance is -0.8 , belonging to $(-\infty, -0.6)$, the power flow can be feasible and all the eigenvalues are still nonnegative; when the value of the reactance is -0.1 , belonging to $(-0.6, 0)$, the power flow can be feasible and there is one negative eigenvalue while the others are nonnegative; when the value of the reactance is exactly -0.6 , the power flow cannot be feasible and there are two zero-eigenvalues, which leads $\mathcal{L}(\mathcal{G}_+)$ to be singular. This example verifies that the negative reactance may lead the DC power flow to be infeasible and the proposed corollary can

compute the exact value of the negative reactance that can make the DC power flow infeasible.

TABLE I. IMPACT OF NEGATIVE REACTANCE ON SIMULATION RESULTS.

Value of NR on Line 2	Eigenvalues of B-matrix	Feasibility of Power Flow
0.1	(0, 6.9514, 15.4410, 32.6076)	Yes
-0.1	(-13.7322, 0, 9.4336, 19.2986)	Yes
-0.6	(0, 0, 10.7644, 10.7644)	No
-0.8	(0, 0.5447, 10.8966, 21.0587)	Yes

However, the location of the negative reactance may also have the impact on the eigenvalues of the B-matrix and DC power flow. Here, we consider two cases: (a) One negative reactance is on line 1 and the other three lines are all with positive reactance, i. e., 0.1 p.u. for line 2, 0.4 p.u. for line 3 and 0.2 p.u. for line 4; (b) Two negative reactances are on line 2 and line 4 and the other two lines are with positive reactance, i. e., 0.1 p.u. for line 1 and 0.4 p.u. for line 3. The above two cases are depicted in the Fig. 6 and Fig. 7, where both the sub-graphs \mathcal{G}_+ are not connected graphs and the \mathbf{B} matrix is always indefinite according to Section IV-A.

Based on the proposed theory in Section IV-B, for the case (a), line 1 is the cut of the network, which gives the fact that the DC power flow can be feasible for any value of negative reactance although the B matrix is always indefinite; for the case (b), we can construct two new graphs according to (36), as

$$\text{shown in Fig. 8 and Fig. 9, leading to } \mathcal{C}_{\mathcal{B}}(\mathcal{G}) = \frac{x_{23}Z_{\mathcal{B}}(\mathcal{G}_{23-})}{x_{23} + Z_{\mathcal{B}}(\mathcal{G}_{23-})}$$

$$\text{and } \mathcal{C}_{\mathcal{A}}(\mathcal{G}) = \frac{x_{34}Z_{\mathcal{A}}(\mathcal{G}_{34-})}{x_{34} + Z_{\mathcal{A}}(\mathcal{G}_{34-})}. \text{ With respect to the circuit theory,}$$

$Z_{\mathcal{B}}(\mathcal{G}_{23-}) = x_{34} + x_{24}$ and $Z_{\mathcal{A}}(\mathcal{G}_{34-}) = x_{23} + x_{24}$. Then, we can conclude that when $x_{23} + Z_{\mathcal{B}}(\mathcal{G}_{23-}) = 0$ or $x_{34} + Z_{\mathcal{A}}(\mathcal{G}_{34-}) = 0$, i.e. $x_{23} + x_{24} + x_{34} = 0$, the DC power flow will be infeasible. Since $x_{24}=0.4$ and both x_{23} and x_{34} are negative values, we will have $|x_{23} + x_{34}| = 0.4$ when DC power flow will be infeasible.

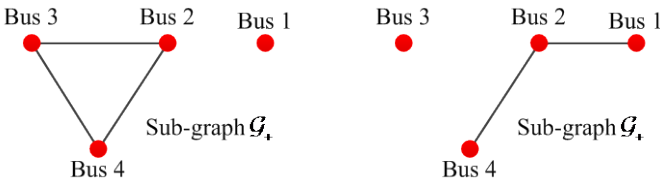


Fig. 6 \mathcal{G}_+ under case (a)

Fig. 7 \mathcal{G}_+ under case (b)

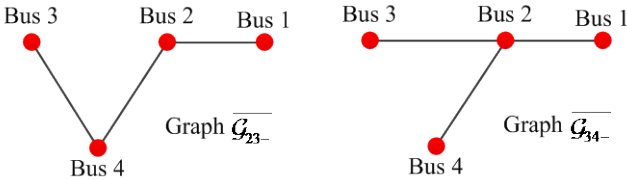


Fig. 8 New graph under case (a) Fig. 9 New graph under case (b)

Furthermore, simulations are performed to compare the above results with those from the proposed theory. For the case (a), we gradually decrease the value of the negative reactance on the line 1 to test DC power flow and compute the eigenvalues of

the \mathbf{B} matrix. Fig. 10 depicts the relationship between the value of the negative reactance and eigenvalues of the \mathbf{B} matrix, where there is one negative eigenvalue. This verifies the viewpoint in Section-A and (37) that the B matrix is always indefinite and DC power flow can be feasible for all the values of the negative reactance.

For case (b), different values of the negative reactance are tested, where it satisfies $|x_{23} + x_{34}| = 0.4$. The eigenvalues of the \mathbf{B} matrix and the feasibility of DC power flow are presented in Table II, where there are two zero-eigenvalues in the \mathbf{B} matrix and the DC power flow cannot be feasible. The conclusion is consistent with the proposed theory.

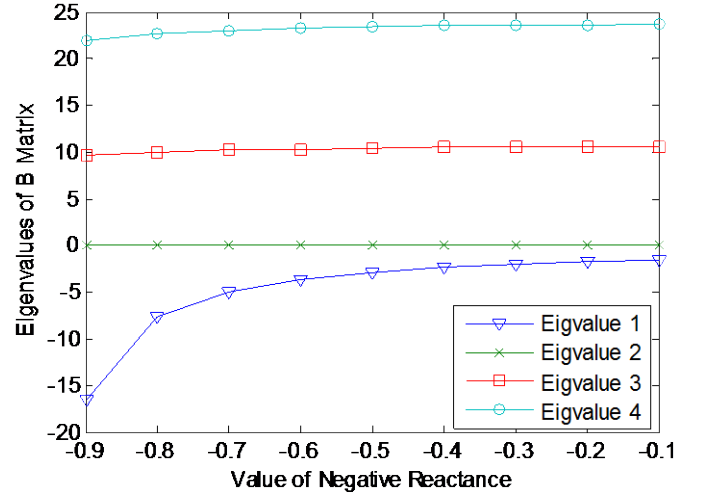


Fig. 10 Relationship between eigenvalues of the \mathbf{B} matrix and the values of the negative reactance

TABLE II. IMPACT OF NEGATIVE REACTANCE ON SIMULATION RESULTS.

Value of NRs	Eigenvalues of B-matrix	Feasibility
(-0.05, -0.35)	(-37.6803, 0, 0, 16.9660)	No
(-0.10, -0.30)	(-19.7814, 0, 0, 18.1147)	No
(-0.15, -0.25)	(-15.1479, 0, 0, 18.8145)	No
(-0.20, -0.20)	(-14.2705, 0, 0, 19.2705)	No
(-0.25, -0.15)	(-15.9157, 0, 0, 19.5824)	No
(-0.30, -0.10)	(-21.4657, 0, 0, 19.7990)	No
(-0.35, -0.05)	(-40.6554, 0, 0, 19.9411)	No

B. IEEE 300-bus System and Real-World Power Systems

As discussed in the introduction, there exists “negative reactance” in real-world system models. For the IEEE 300-bus system, the branch #1201-#120 is a NR with the value of -0.3697. It is investigated from the topology analysis that \mathcal{G}_+ is a connected graph and we have $Z(\mathcal{G}_+) = 0.6619$ with respect to (9). Moreover, Table III gives three cases with different value of NR, where the DC power flow is infeasible when the value of NR is exact -0.6619. With the increase/decrease of the absolute value of NR, one zero-eigenvalue will become positive/negative. The phenomenon is consistent with the analysis in Fig. 3, which verifies the proposed theory.

TABLE III. NEGATIVE REACTANCE ON POWER FLOW FEASIBILITY.

Value of NR	Eigenvalues of B-matrix	Feasibility
-0.6630	(0, 0.002841, 0.07978, 0.09206, ...)	Yes
-0.6619	(0, 0, 0.07978, 0.09206, ...)	No
-0.6610	(-0.002244, 0, 0.07978, 0.09206, ...)	Yes

As for three real-life Polish power systems, the 3012-/3120-bus system has 10 NRs and 3375-bus system has 12 NRs. From the topology analysis, we can find that all the NRs are resulted from the mutual effects between windings of the three-winding transformers. Particularly, the transformer is at the end of the system, which connects the power generator and the system. If the branch is removed from the system, the power generator will be isolated. Therefore, the NR on the transformer is always the cut of the system. We randomly choose the value of NRs and run the DC power flow. We conclude that the DC power flow can always be feasible while the B matrix is indefinite, which is consistent with (37) and discussion in Section IV-A.

The simulation results suggest that the location and the value of the NR may affect the feasibility of DC power flow, which is a new phenomenon that hasn't been explored in literature. The infeasibility of DC power flow implies that the power cannot transfer from one part to another part.

VI. CONCLUSIONS

The value of the negative reactance may have impacts on the definiteness of the B matrix in DC power flow equations, which will subsequently affect the feasibility of DC power flow. This paper introduces the concept of the effective resistance and gives the corollaries to investigate the eigenvalues of the B matrix with negative reactances. The results on several test systems have verified the proposed corollaries and theorems. Most importantly, it concludes that

- (i) The definiteness of Laplacian matrix for the graph will be affected by the negative weights. Meanwhile, the sufficient and necessary condition for the positive definiteness is derived.
- (ii) The negative reactance may result in "physical dis-connectivity" and lead the linear system to be singular, so that the DC power flow will be infeasible.
- (iii) The relationship between geometric dis-connectivity and physical dis-connectivity is revealed.

Finally, it should be noted that this paper only chooses DC power flow for discussion, which is a linearized approximation of the practical AC power flow. How to extend the conclusions to the AC power flow should be investigated in the future work.

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Tao Ding (S'13–M'15) received the degree in mathematics from Southeast University, Nanjing, China, the B.S.E.E. and M.S.E.E. degrees from Southeast University, Nanjing, China, and the Ph.D. degree from Tsinghua University, Beijing, China, in 2007, 2009, 2012 and 2015, respectively. From 2013 to 2014, he was a Visiting Scholar at the Department of Electrical Engineering and Computer Science, University of Tennessee, Knoxville, TN, USA. He is currently an Associate Professor with the State Key Laboratory of Electrical Insulation and Power Equipment and the School of Electrical Engineering, Xi'an Jiaotong University, Xi'an, China. His research interests include electricity markets, power system economics and optimization methods, and power system planning and reliability evaluation. He was the recipient of the Outstanding Graduate Award of Beijing City.

Rui Bo (SM'10) received the BSEE and MSEE degrees in electric power engineering from Southeast University (China) in 2000 and 2003, respectively, and received the Ph.D. degree from the University of Tennessee, Knoxville (UTK) in 2009. He is presently an assistant professor of Electrical and Computer Engineering Department at Missouri University of Science and Technology, Rolla MO. From 2009 to 2017, he worked at Midcontinent Independent Transmission System Operator (MISO) as a principal engineer and project manager. His interests include power system operation and planning, power system economics, computational methods and high performance computing.

Yongheng Yang (S'12 - M'15) received the B.Eng. degree in Electrical Engineering and Automation from Northwestern Polytechnical University, Xi'an, China, in 2009, and the Ph.D. degree in Electrical Engineering from Aalborg University, Aalborg, Denmark, in 2014. He was a postgraduate at Southeast University, China, from 2009 to 2011. In 2013, he was a Visiting Scholar at Texas A&M University, USA. Since 2014, he has been with the Department of Energy Technology, Aalborg University, where currently he is an Assistant Professor. His research is focused on grid integration of renewable energy systems, power converter design, analysis and control, harmonics identification and mitigation, and reliability in power electronics. Dr. Yang has published more than 80 technical papers and co-authored a book – Periodic Control of Power Electronic Converters (London, UK: IET, 2017). Dr. Yang is a Member of the IEEE Power Electronics Society (PELS) Students and Young Professionals Committee. He served as a Guest Associate Editor of IEEE JOURNAL OF EMERGING AND SELECTED TOPICS IN POWER ELECTRONICS, and has also been invited as a Guest Editor of Applied Sciences. He is an active reviewer for relevant top-tier journals.

Frede Blaabjerg (S'86 - M'88 - SM'97 - F'03) was with ABB-Scandia, Randers, Denmark, from 1987 to 1988. From 1988 to 1992, he was a Ph.D. Student with Aalborg University, Aalborg, Denmark. He became an Assistant Professor in 1992, Associate Professor in 1996, and Full Professor of power electronics and drives in 1998. His current research interests include power electronics and its applications such as in wind turbines, PV systems, reliability, harmonics and adjustable speed drives. He has received 18 IEEE Prize Paper Awards, the IEEE PELS Distinguished Service Award in 2009, the EPE-PEMC Council Award in 2010, the IEEE William E. Newell Power Electronics Award 2014 and the Villum Kann Rasmussen Research Award 2014. He was an Editor-in-Chief of the IEEE TRANSACTIONS ON POWER ELECTRONICS from 2006 to 2012. He was nominated in 2014, 2015 and 2016 by Thomson Reuters to be between the most 250 cited researchers in Engineering in the world.